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REFLECTION AND TRANSMISSION OF PLANE COMPRESSIONAL WAVES†

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The reflected and transmitted waves generated by a plane, monochromatic, compressional wave incident at a plane interface between two half-spaces are analyzed. Energy ratios and the phase angles for the vertical displacement have been tabulated for over two thousand choices of the elastic parameters and densities. Computations for the solid-fluid and solid-air cases show that a large fraction of the incident energy goes into the shear wave over a large range of angles of incidence. The conditions for perfect reflection (with no conversion and no transmission) at the critical angles are derived. Within the critical angle, the requirement that the instantaneous energy flux be continuous is given by Knott's equation. Beyond the critical angle, Knott's equation gives continuity of the net flux but not continuity of the instantaneous flux. To achieve continuity of the instantaneous flux, Knott's equation and two additional equations must be satisfied.

INTRODUCTION

Numerical data on the plane-wave reflection and transmission coefficients are useful in geophysical applications, seismic model studies, and velocity measurements. We have studied the reflection and transmission of a plane, monochromatic P (compressional) wave at a plane interface between two half-spaces. We have systematically varied all of the pertinent parameters of the two media over as wide a range of values as one is likely to encounter in either seismic or modeling applications. For each combination we have tabulated the energy flux ratio in the vertical direction and the relative phase of the vertical displacement in each of the reflected and transmitted waves. This work complements a recent extensive study by Koefoed (1962). Koefoed lists amplitude coefficients while we list energy coefficients, and our phase definitions are different. Costain, Cook, and Algermissen (1963) have tabulated amplitude coefficients, energy ratios, and phase angles for an incident SV (shear wave polarized in the vertical plane through the source and receiver) for media which have a Poisson's ratio of 0.25.

The purpose of this paper is to draw attention to the tabulated plane-wave data, to analyze the effect of the parameters on the way in which the energy is partitioned among the phases, to state the conditions for perfect reflection (with no conversion) at the critical angles, and to clarify the physical significance of Knott's equation beyond the critical angle.

TABULATED DATA

The parameters which we have chosen to characterize the media are the P-wave-velocity ratio, $V21$; the density ratio, $R021$; and the Poisson's ratios of the two media, $SIG1$ and $SIG2$. The notation $V21$ represents the velocity in medium 2 divided by the velocity in medium 1, similarly with $R021$. The incident P wave approaches the interface through medium 1.

We have considered the following range of parameters:

- $V21$—0.25, 0.50, 0.75, 1.00, 1.50, 2.0, 3.0, and 4.0;
- $R021$—0.33, 0.50, 0.80, 0.90, 1.00, 1.10, 1.20, 1.50, 2.00, and 3.00;
- $SIG1$—0.10, 0.20, 0.25, 0.30, 0.40, and 0.50;
- $SIG2$—0.10, 0.20, 0.25, 0.30, 0.40, and 0.50.

A material with a Poisson's ratio of 0.50 is a fluid. We have considered all combinations of the above parameters except those where the fluid velocity exceeds the P-wave velocity in the solid or where

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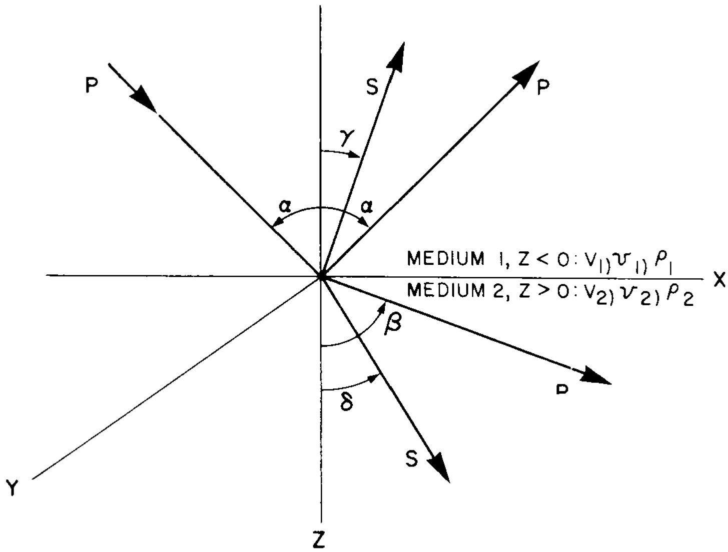

FIG. 1. Angles which the incident, reflected, and transmitted waves make with the normal to the interface.

the fluid density exceeds the density in the solid. In addition, we have considered the cases where $V21=0.10$, $R021=0.0005$, $SIG2=0.5$, and $SIG1$ varies from 0.10 to 0.50 as given above. This set of cases corresponds to a wave incident upon an air boundary from a solid or liquid. And, conversely, we have treated the air wave incident upon a solid or liquid boundary. Here $V21=10$, $R021=2{,}000$, $SIG1=0.5$, and $SIG2$ varies from 0.10 to 0.50.

For each of the some two thousand different combinations of parameters, we have computed the energy-flux ratios and relative phases at five-degree intervals for angles of incidence ranging from 0 to 85 degrees. When a critical angle is present, we also calculate the energy partition at the critical angle and at one-degree intervals within five degrees of the critical angle. All this data is published in a VELA UNIFORM report (Tooley, Spencer, Sagoci, 1963).

REFLECTION AND TRANSMISSION COEFFICIENTS

The angles which the incident, reflected, and transmitted waves make with the normal to the interface are indicated in Figure 1. The cartesian coordinate system is so oriented that the XY plane coincides with the interface and the positive Z axis points downward. The incident wave approaches the interface through medium 1. The normal to the incident wavefront lies in the XZ plane and makes an angle α with the Z axis. The phase velocity along the interface is

$$c = V_1/\sin\alpha. \tag{1}$$

V_i, v_i, and ρ_i are the compressional velocity, shear velocity, and density in the ith medium, respectively. The vertical and horizontal displacements in the incident wave are

$$\begin{aligned} w_0 &= G_0 \cos\psi, \\ \psi &= \omega t - \frac{\omega}{V_1}(x\sin\alpha + z\cos\alpha), \\ u_0 &= G_0 \tan\alpha\cos\psi, \end{aligned} \tag{2}$$

where G_0 is an arbitrary positive quantity which determines the amplitude of the vertical displacement. The wavenumber is

$$k = \omega/c = \omega\sin\alpha/V_1. \tag{3}$$

Let G_+ and G_- represent the amplitudes of the vertical displacement in the reflected P and S waves, respectively. The displacements in medium 1 are

(Please turn the page for equation 4)

$$w_1 = \Re\{e^{i(\omega t-kx)}[G_0 e^{-p_+ z} + G_+ e^{p_+ z} + G_- e^{p_- z}]\}, \qquad z<0, \tag{4}$$

$$u_1 = \Re\left\{e^{i(\omega t-kx)}\left[G_0(\tan\alpha)e^{-p_+ z} + \frac{\omega \sin\alpha}{iV_1}\frac{G_+}{p_+}e^{p_+ z} + \frac{\omega}{iV_1 \sin\alpha}\left(\sin^2\alpha - \frac{V_1^2}{v_1^2}\right)\frac{G_-}{p_-}e^{p_- z}\right]\right\},$$

where

$$p_+ = i\omega \sin\alpha / V_1, \qquad p_- = (i\omega/V_1)(V_1^2/v_1^2 - \sin^2\alpha)^{1/2}. \tag{5}$$

The symbol $\Re\{\ \}$ means that only the real part of the quantity in parentheses is required.

Let H_+ and H_- represent the amplitudes of the vertical displacement in the transmitted P and S waves, respectively. The displacements in medium 2 are

$$w_2 = \Re\{e^{i(\omega t-kx)}[H_+ e^{q_+ z} + H_- e^{q_- z}]\}, \qquad z>0, \tag{6}$$

$$u_2 = \Re\left\{e^{i(\omega t-kx)}\left[\frac{\omega \sin\alpha}{iV_1}\frac{H_+}{q_+}e^{q_+ z} + \frac{\omega}{iV_1\sin\alpha}\left(\sin^2\alpha - \frac{V_1^2}{v_2^2}\right)\frac{H_-}{q_-}e^{q_- z}\right]\right\},$$

where

$$\begin{aligned} q_+ &= -(i\omega/V_1)(V_1^2/V_2^2 - \sin^2\alpha)^{1/2}, \qquad \sin\alpha < V_1/V_2, \\ q_- &= -(i\omega/V_1)(V_1^2/v_2^2 - \sin^2\alpha)^{1/2}, \qquad \sin\alpha < V_1/v_2. \end{aligned} \tag{7}$$

Beyond the critical angle for P,

$$q_+ = -(\omega/V_1)(\sin^2\alpha - V_1^2/V_2^2)^{1/2}, \qquad \sin\alpha > \sin\alpha_P = V_1/V_2. \tag{8}$$

Beyond the critical angle for S,

$$q_- = -(\omega/V_1)(\sin^2\alpha - V_1^2/v_2^2)^{1/2}, \qquad \sin\alpha > \sin\alpha_S = V_1/v_2. \tag{9}$$

These relations show how the square roots of negative numbers must be interpreted.

Comparison between the horizontal and vertical displacements in the individual waves indicates that: In the reflected P and S waves the displacements are 180 degrees out-of-phase and the particle motion is linear. In the transmitted P and S waves the displacements are in-phase inside the critical angles and the particle motion is linear. Beyond the critical angles, the displacements are 90 degrees out-of-phase and the motion is elliptical.

We assume that the two media are perfectly coupled together at the interface. Consequently, the displacements and the normal and tangential stresses must be continuous across the interface. This requirement leads to four linear simultaneous equations which can be solved for

$$R_{PP} = \frac{G_+}{G_0}, \qquad R_{PS} = \frac{G_-}{G_0},$$

$$T_{PP} = \frac{H_+}{G_0}, \qquad T_{PS} = \frac{H_-}{G_0}. \tag{10}$$

These quantities are the reflection and transmission coefficients for the vertical displacement in the incident compressional wave. It will be shown that for $\sin\alpha < V_1/V_2$ the coefficients are all real. Equations (4) and (6) show that at the interface the coefficients determine the ratio of the vertical displacement in each of the reflected and transmitted waves to the vertical displacement in the incident wave. Beyond the compressional critical angle the coefficients are complex—

$$R_{PP} = A_{1P}e^{i\phi_{1P}}, R_{PS} = A_{1S}e^{i\phi_{1S}}, T_{PP} = A_{2P}e^{i\phi_{2P}}, T_{PS} = A_{2S}e^{i\phi_{2S}}, \ \sin\alpha > V_1V_2. \tag{11}$$

The A_{ij} are the amplitude factors and the ϕ_{ij} are the phase angles for the vertical displacement. The fact that the coefficients are complex indicates that the reflected and transmitted waves are out-of-phase with the incident wave. Consequently, the coefficients do not determine the ratio of the vertical displacements—in fact, that ratio is time-dependent. The ϕ_{ij} satisfy

$$0 \le \phi_{ij} < 2\pi$$

and are listed in the tables.

The reflection and transmission coefficients which are written out below are expressed as functions of $\sin\theta$. θ is the acute angle between the Z axis and the normal to the wavefront which travels with velocity V_r. V_r may be set equal to any of the body-wave velocities (V_i, v_i). The choice of V_r determines the physical significance of θ. In Figure 2A, $V_r = V_1$ and $\theta = \alpha$ (the angle of the incident P wave). In Figure 2B, $V_r = v_2$ and θ is the angle of the transmitted S wave.

The coefficients contain the parameter

$$\frac{x}{\rho_r V_r^2} = \frac{\rho_1 v_1^2}{\rho_r V_r^2} - \frac{\rho_2 v_2^2}{\rho_r V_r^2}.$$

ρ_r has the units of density and could be the density in either medium. The coefficients depend on the four square roots

$$\begin{aligned} a_1 &= (-(V_1/V_r)^2 \sin^2\theta + 1)^{1/2}, & b_1 &= (-(v_1/V_r)^2 \sin^2\theta + 1)^{1/2}, \\ a_2 &= (-(V_2/V_r)^2 \sin^2\theta + 1)^{1/2}, & b_2 &= (-(v_2/V_r)^2 \sin^2\theta + 1)^{1/2}, \end{aligned} \tag{12}$$

and the functions

$$l = \frac{x}{\rho_r V_r^2}\sin^2\theta + \frac{1}{2}\left(\frac{\rho_2}{\rho_r} - \frac{\rho_1}{\rho_r}\right), \quad p = \frac{x}{\rho_r V_r^2}\sin^2\theta - \frac{1}{2}\frac{\rho_1}{\rho_r}, \quad q = \frac{x}{\rho_r V_r^2}\sin^2\theta + \frac{1}{2}\frac{\rho_2}{\rho_r}.$$

The reflection and transmission coefficients are all of the form A/D. The denominator is

$$\begin{aligned} D = &-\left(\frac{V_1 v_1 V_2 v_2}{V_r^4}\right) l^2 \sin^2\theta - \frac{x^2}{\rho_r^2 V_r^4}\sin^2\theta\, a_1 b_1 a_2 b_2 - \frac{v_1}{V_r} b_2 \left\{\frac{V_1}{V_r} a_2 p^2 + \frac{1}{4}\left(\frac{\rho_1\rho_2}{\rho_r^2}\right)\frac{V_2}{V_r} a_1\right\} \\ &- \frac{v_2}{V_r} b_1 \left\{\frac{V_2}{V_r} a_1 q^2 + \frac{1}{4}\left(\frac{\rho_1\rho_2}{\rho_r^2}\right)\frac{V_1}{V_r} a_2\right\}. \end{aligned} \tag{13}$$

The denominator is the same for all coefficients associated with a particular interface. It is readily verified that interchanging the subscripts 1 and 2 does not alter the expression for D in any way. This

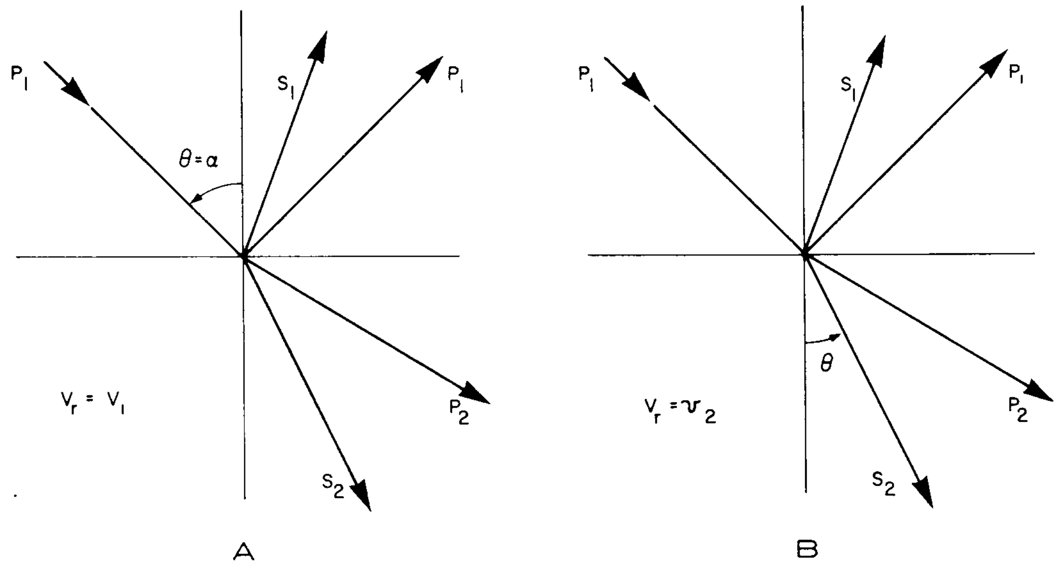

FIG. 2. The choice of V_r determines the physical significance of θ. θ is the angle between the normal to the interface and the normal to the wavefront which propagates with velocity V_r.

shows that the denominator is the same regardless of whether the incident wave approaches the interface from above or below.

The numerators for the reflection and transmission coefficients for an incident compressional wave which approaches the interface through medium 1 can be put in the form:

$$DR_{PP} = -\left(\frac{V_1 v_1 V_2 v_2}{V_r^4}\right) l^2 \sin^2\theta + \frac{x^2}{\rho_r^2 V_r^4}\sin^2\theta\, a_1 b_1 a_2 b_2 - \frac{v_1}{V_r} b_2 \left\{\frac{V_1}{V_r} a_2 p^2 - \frac{1}{4}\left(\frac{\rho_1\rho_2}{\rho_r^2}\right)\frac{V_2}{V_r} a_1\right\}$$
$$+ \frac{v_2}{V_r} b_1 \left\{\frac{V_2}{V_r} a_1 q^2 - \frac{1}{4}\left(\frac{\rho_1\rho_2}{\rho_r^2}\right)\frac{V_1}{V_r} a_2\right\}, \tag{14}$$

$$DR_{PS} = 2\left(\frac{V_1 v_1}{V_r^2}\right)\sin^2\theta \left\{\left(\frac{V_2 v_2}{V_r^2}\right) lq + \frac{x}{\rho_r V_r^2} p a_2 b_2\right\}, \tag{15}$$

$$DT_{PP} = -\frac{\rho_1 V_1}{\rho_r V_r} a_2 \left\{\frac{v_2}{V_r} b_1 q - \frac{v_1}{V_r} b_2 p\right\}, \tag{16}$$

$$DT_{PS} = \left(\frac{v_2 V_1}{V_r^2}\right)\frac{\rho_1}{\rho_r}\sin^2\theta \left\{\left(\frac{V_2 v_1}{V_r^2}\right) l + \frac{x}{\rho_r V_r^2} a_2 b_1\right\}. \tag{17}$$

The corresponding expressions for the reflection and transmission coefficients for an incident shear wave polarized in the XZ plane (i.e., an SV wave) are:

$$DR_{SS} = \left(\frac{V_1 v_1 V_2 v_2}{V_r^4}\right) l^2 \sin^2\theta - \frac{x^2}{\rho_r^2 V_r^4}\sin^2\theta\, a_1 b_1 a_2 b_2 + \frac{v_1}{V_r} b_2 \left\{\frac{V_1}{V_r} a_2 p^2 + \frac{1}{4}\left(\frac{\rho_1\rho_2}{\rho_r^2}\right)\frac{V_2}{V_r} a_1\right\}$$
$$- \frac{v_2}{V_r} b_1 \left\{\frac{V_2}{V_r} a_1 q^2 + \frac{1}{4}\left(\frac{\rho_1\rho_2}{\rho_r^2}\right)\frac{V_1}{V_r} a_2\right\}, \tag{18}$$

$$DR_{SP} = 2 a_1 b_1 \left\{\left(\frac{V_2 v_2}{V_r^2}\right) lq + \frac{x}{\rho_r V_r^2} p a_2 b_2\right\}, \tag{19}$$

$$DT_{SP} = -\frac{\rho_1}{\rho_r} a_2 b_1 \left\{\left(\frac{V_1 v_2}{V_r^2}\right) l + \frac{x}{\rho_r V_r^2} a_1 b_2\right\}, \tag{20}$$

$$DT_{SS} = -\frac{\rho_1}{\rho_r}\frac{v_2}{V_r} b_1 \left\{\frac{V_2}{V_r} a_1 q - \frac{V_1}{V_r} a_2 p\right\}. \tag{21}$$

In this form the coefficients are perfectly general and can be easily specialized to treat the fluid-solid, fluid-fluid, and solid-vacuum cases. For the case where the second medium is a vacuum, the transmission coefficients are not all zero. This does not mean that motion is induced in the vacuum. This is impossible, because in the limit as the velocities go to zero it takes an infinite time to convey the information that the interface has moved to any point which is at a finite distance from the interface. Therefore, the particular values approached by the transmission coefficients are unimportant.

The coefficients for the liquid-liquid case are

$$R_{PP} = \frac{a_2 \dfrac{\rho_1 V_1}{\rho_r V_r} - a_1 \dfrac{\rho_2 V_2}{\rho_r V_r}}{a_2 \dfrac{\rho_1 V_1}{\rho_r V_r} + a_1 \dfrac{\rho_2 V_2}{\rho_r V_r}},$$

$$T_{PP} = \frac{2a_2 \dfrac{\rho_1 V_1}{\rho_r V_r}}{a_2 \dfrac{\rho_1 V_1}{\rho_r V_r} + a_1 \dfrac{\rho_2 V_2}{\rho_r V_r}}. \tag{22}$$

For $V_1/V_2=1$, the coefficients do not depend on the angle of incidence.

The square roots in equation (12) vanish at critical angles and at grazing incidence ($\theta=\pi/2$). When θ is less than the smallest of the critical angles, the square roots are all real. Consequently, the reflection and transmission coefficients are also real and the phase shift with respect to the incident wave is either zero or π. When θ is greater than the smallest of the critical angles, at least one of the square roots is negative imaginary. Consequently, all the coefficients are, in general, complex.

Now we consider the form of the compressional coefficients for certain special values of θ. Let $V_r = V_1$ so that θ is the direction of the incident compressional wave. a_1 vanishes at $\theta=\pi/2$. b_1 cannot vanish for any θ. If $V_1/V_2>1$, neither a_2 nor b_2 can vanish, there are no critical angles, and the transmitted waves are either in-phase with or 180 degrees out-of-phase with the incident wave. If $V_1/V_2<1$, a_2 is negative imaginary for $\theta>\alpha_P=\sin^{-1} V_1/V_2$. At α_P the coefficients reduce to

$$
\begin{aligned}
D &= -\frac{V_2}{V_1} D_{PP},\\
D_{PP} &= \frac{v_1v_2}{V_2^2}\left(K_{PP}-\frac{1}{2}\right)^2 + \left(1-\frac{V_1^2}{V_2^2}\right)^{1/2}\\
&\quad\cdot\left\{\frac{1}{4}\,\frac{\rho_2 v_1}{\rho_1 V_1}\left(1-\frac{v_2^2}{V_2^2}\right)^{1/2} + \frac{v_2}{V_1}\left(1-\frac{v_1^2}{V_2^2}\right)^{1/2} K_{PP}^2\right\}, \qquad (23)\\
R_{PP}D_{PP} &= \frac{v_1v_2}{V_2^2}\left(K_{PP}-\frac{1}{2}\right)^2 - \left(1-\frac{V_1^2}{V_2^2}\right)^{1/2}\\
&\quad\cdot\left\{\frac{1}{4}\,\frac{\rho_2 v_1}{\rho_1 V_1}\left(1-\frac{v_2^2}{V_2^2}\right)^{1/2} + \frac{v_2}{V_1}\left(1-\frac{v_1^2}{V_2^2}\right)^{1/2} K_{PP}^2\right\},\\
R_{PS}D_{PP} &= -2\frac{v_1v_2}{V_2^2} K_{PP}\left(K_{PP}-\frac{1}{2}\right),\\
T_{PP} &= 0,\\
T_{PS}D_{PP} &= -\frac{v_1v_2}{V_2^2}\left(K_{PP}-\frac{1}{2}\right),
\end{aligned}
$$

where

$$K_{PP} = \frac{v_1^2}{V_2^2} - \frac{\rho_2}{\rho_1}\,\frac{v_2^2}{V_2^2} + \frac{1}{2}\,\frac{\rho_2}{\rho_1}\,. \qquad (24)$$

There is no vertical displacement in the transmitted P wave. Note that when

$$K_{PP} = \frac{1}{2} \qquad (25)$$

the coefficients reduce to

$$R_{PP} = -1, \quad R_{PS} = 0, \quad T_{PP} = 0, \quad T_{PS} = 0 \quad (\theta = \alpha_P).$$

Equation (25) is the condition for perfect reflection at the critical angle with no conversion and no transmission.

If $V_1/v_2<1$, b_2 becomes negative imaginary for $\theta>\alpha_s=\sin^{-1} V_1/v_2$. α_s is the critical angle for the transmitted shear wave. At α_s the coefficients are complex and take the form

$$D = -\frac{v_2}{V_1} D_{PS},$$

$$D_{PS} = \frac{v_1 V_2}{v_2^2}\left(K_{PS} - \frac{1}{2}\right)^2 + \left(1 - \frac{v_1^2}{v_2^2}\right)^{1/2} \cdot \left\{\frac{V_2}{V_1}\left(1 - \frac{V_1^2}{v_2^2}\right)^{1/2} K_{PS}^2 - \frac{i}{4}\frac{\rho_2}{\rho_1}\left(\frac{V_2^2}{v_2^2} - 1\right)^{1/2}\right\}, \tag{26}$$

$$R_{PP}D_{PS} = \frac{v_1 V_2}{v_2^2}\left(K_{PS} - \frac{1}{2}\right)^2 - \left(1 - \frac{v_1^2}{v_2^2}\right)^{1/2} \cdot \left\{\frac{V_2}{V_1}\left(1 - \frac{V_1^2}{v_2^2}\right)^{1/2} K_{PS}^2 + \frac{i}{4}\frac{\rho_2}{\rho_1}\left(\frac{V_2^2}{v_2^2} - 1\right)^{1/2}\right\},$$

$$R_{PS}D_{PS} = -2\frac{v_1 V_2}{v_2^2} K_{PS}\left(K_{PS} - \frac{1}{2}\right),$$

$$T_{PP}D_{PS} = -i\left(\frac{V_2^2}{v_2^2} - 1\right)^{1/2}\left(1 - \frac{v_1^2}{v_2^2}\right)^{1/2} K_{PS},$$

$$T_{PS}D_{PS} = -\frac{v_1 V_2}{v_2^2}\left(K_{PS} - \frac{1}{2}\right) + i\left(\frac{v_1^2}{v_2^2} - \frac{\rho_2}{\rho_1}\right)\left(\frac{V_2^2}{v_2^2} - 1\right)^{1/2}\left(1 - \frac{v_1^2}{v_2^2}\right)^{1/2},$$

where

$$K_{PS} = \frac{v_1^2}{v_2^2} - \frac{1}{2}\frac{\rho_2}{\rho_2}. \tag{27}$$

In the special case

$$K_{PS} = 0 \tag{28}$$

the coefficients reduce to

$$R_{PP}=1, \quad R_{PS}=0, \quad T_{PP}=0, \quad T_{PS}=2. \tag{29}$$

At or beyond the critical angle, α_S, the vertical energy flux in the transmitted waves vanishes when integrated over a period. Consequently when $\theta \geq \alpha_S$, $R_{PS}=0$ is the condition for perfect reflection. This condition is satisfied when either equation (28) holds or $K_{PS}=\frac{1}{2}$. In the latter case the reflected compressional wave experiences a phase shift. Equation (29) shows that the coefficient for a particular wave may exceed unity beyond the critical angle for that wave.

At grazing incidence $\theta=\pi/2$, a_1 vanishes and $R_{PP}=1$. This condition is always satisfied regardless of the values of the velocity ratios.

We can treat special cases for an incident shear wave in similar fashion by setting $V_r=v_1$ in equations (18)–(21). θ is then the angle between the vertical and the normal to the wavefront of the incident shear wave. Because $v_1 < V_1$, there is always a critical angle for the reflected compressional wave. This angle is defined by $\beta_{SP} = \sin^{-1} v_1/V_1$. β_{SP} is the value of θ which causes a_1 to vanish. b_1 vanishes at grazing incidence. a_2 vanishes at $\beta_P = \sin^{-1} v_1/V_2$ when $v_1/V_2 < 1$ and does not vanish otherwise. b_2 vanishes at $\beta_S = \sin^{-1} v_1/v_2$ when $v_1/v_2 < 1$ and does not vanish otherwise. β_P and β_S are the critical angles for the transmitted compressional and shear waves, respectively. For an incident shear wave there is always one critical angle and there may be as many as three, whereas for an incident compressional wave a critical angle exists only if $V_1/V_2 < 1$, and there are never more than two.

ENERGY FLUX RATIOS AND KNOTT'S EQUATION

The energy flux normal to the interface (Love, 1944) is determined by substituting equations (4) and (6) for the displacements in

$$F_z = -\left(\frac{\partial u}{\partial t}\tau_{XZ} + \frac{\partial w}{\partial t}\tau_{XZ}\right),$$

where τ_{XZ} and τ_{XZ} are the shearing stress and the vertical component of the normal stress, respec-

tively. Let $\phi = \omega t - kx$. The vertical flux in each wave type is then given by:
Incident P

$$F_0 = \frac{\omega^2 G_0^2 \rho_1 V_1 \sin^2 \psi_0}{\cos\alpha}, \qquad \psi_0 = \phi - \frac{z\,\omega\cos\alpha}{V_1}, \tag{30}$$

Reflected P

$$F_{1P} = -\frac{\omega^2 G_0^2 \rho_1 V_1 A_{1p}^2 \sin^2 \psi_{1P}}{\cos\alpha}, \qquad \psi_{1P} = \phi + \frac{z\,\omega\cos\alpha}{V_1} + \phi_{1P}, \tag{31}$$

Reflected S

$$F_{1S} = -\frac{\omega^2 G_0^2 \rho_1 V_1 A_{1S}^2}{\sin^2\alpha}\left(\frac{V_1^2}{v_1^2} - \sin^2\alpha\right)^{1/2} \sin^2 \psi_{1S},$$

$$\psi_{1S} = \phi + \frac{\omega}{V_1}\left(\frac{V_1^2}{v_1^2} - \sin^2\alpha\right)^{1/2} z + \phi_{1S}, \tag{32}$$

Transmitted P

$$\alpha \le \sin^{-1} V_1/V_2,$$

$$F_{2P} = \omega^2 G_0^2 \rho_2 V_1 A_{2P}^2 \left(\frac{V_1^2}{V_2^2} - \sin^2\alpha\right)^{-1/2} \sin^2 \psi_{2P},$$

$$\psi_{2P} = \phi - \frac{\omega}{V_1}\left(\frac{V_1^2}{V_2^2} - \sin^2\alpha\right)^{1/2} z + \phi_{2P}, \tag{33}$$

$$\alpha > \sin^{-1} V_1/V_2,$$

$$F_{2P} = \frac{\omega^2 G_0^2 \rho_2 V_1 A_{2P}^2 e^{-2(\omega/V_1)(\sin^2\alpha - V_1^2/V_2^2)^{1/2} z}}{\left(\sin^2\alpha - \dfrac{V_1^2}{V_2^2}\right)^{1/2}} \sin\psi_{2P}\cos\psi_{2P}, \quad \psi_{2P} = \phi + \phi_{2P}, \tag{34}$$

Transmitted S

$$\alpha \le \sin^{-1} V_1/v_2$$

$$F_{2S} = \frac{\omega^2 G_0^2 \rho_2 V_1 A_{2S}^2}{\sin^2\alpha}\left(\frac{V_1^2}{v_2^2} - \sin^2\alpha\right)^{1/2} \sin^2 \psi_{2S},$$

$$\psi_{2S} = \phi - \frac{\omega}{V_1}\left(\frac{V_1^2}{v_2^2} - \sin^2\alpha\right)^{1/2} z + \phi_{2S}, \tag{35}$$

$$\alpha > \sin^{-1} V_1/v_2$$

$$F_{2S} = -\frac{\omega^2 G_0^2 \rho_2 V_1 A_{2S}^2}{\sin^2\alpha}\left(\sin^2\alpha - \frac{V_1^2}{v_2^2}\right)^{1/2} e^{-2(\omega/V_1)(\sin^2\alpha - V_1^2/v_2^2)^{1/2} z} \sin\psi_{2S}\cos\psi_{2S},$$

$$\psi_{2S} = \phi + \phi_{2S}. \tag{36}$$

Inside the critical angle for a particular wave, the time and x dependence always enter in the form

$$\sin^2(\omega t - kx + \phi_{ij}).$$

This means that the flux *direction* (not the magnitude) is independent of time and position on the interface. For the incident and transmitted waves, the flux is always in the positive Z direction while for the reflected waves the flux is always in the negative Z direction. This unidirectional property of the flux is related to the fact that the vertical and horizontal displacements in each wave are either in-phase or 180 degrees out-of-phase.

Beyond the critical angle for a particular wave the time and x dependence enter in the form

$$\sin(\omega t - kx + \phi_{ij})\cos(\omega t - kx + \phi_{ij}).$$

The flux direction depends on both the time and position. At a particular x the flux direction reverses every quarter cycle. At a particular time the flux direction reverses every quarter wavelength along the interface. At each point there is no net transfer of energy across the interface in a half cycle, and the instantaneous flux across a half-wavelength section of the interface is zero. The fact that the flux alternates in direction is related to the fact that the horizontal and vertical displacements are 90 degrees out-of-phase.

The instantaneous energy flux normal to the interface must be continuous across the interface, otherwise the energy density at the interface would become infinite. Inside the critical angle for P the requirement that the flux component normal to the interface be continuous is

$$\frac{\omega^2 G_0^2 \rho_1 V_1 \sin^2\phi}{\cos\alpha} - \frac{\omega^2 G_0^2 \rho_1 V_1 A_{1P}^2 \sin^2(\phi + \phi_{1P})}{\cos\alpha} - \frac{\omega^2 G_0^2 \rho_1 V_1 A_{1S}^2}{\sin^2\alpha}\left(\frac{V_1^2}{v_1^2} - \sin^2\alpha\right)^{1/2} \sin^2(\phi + \phi_{1S})$$

$$= \frac{\omega^2 G_0^2 \rho_2 V_1 A_{2p}^2}{\left(\frac{V_1^2}{V_2^2} - \sin^2\alpha\right)^{1/2}} \sin^2(\phi + \phi_{2P}) + \frac{\omega^2 G_0^2 \rho_2 V_1 A_{2S}^2}{\sin^2\alpha}\left(\frac{V_1^2}{v_2^2} - \sin^2\alpha\right)^{1/2} \sin^2(\phi + \phi_{2S}),$$

$$\alpha \leq \alpha_P = \sin^{-1} V_1/V_2 \tag{37}$$

Inside the critical angle for P all the ϕ_{ij} are either zero or π. This means that the time and x dependence enter into the flux component for each wave in exactly the same way. We can find the instantaneous partition of the incident flux by dividing each term in equation (37) by the first term. In the resulting equation a minus sign indicates that the incident flux and the flux in the wave under consideration are in opposite directions. Rearranging terms gives Knott's equation (Knott, 1899)

$$1 = A_{1P}^2 + A_{1S}^2 \frac{\cos\alpha}{\sin^2\alpha}\left(\frac{V_1^2}{v_1^2} - \sin^2\alpha\right)^{1/2} + A_{2P}^2 \frac{\rho_2}{\rho_1} \frac{\cos\alpha}{\left(\frac{V_1^2}{V_2^2} - \sin^2\alpha\right)^{1/2}}$$

$$+ A_{2S}^2 \frac{\rho_2}{\rho_1} \frac{\cos\alpha}{\sin^2\alpha}\left(\frac{V_1^2}{v_2^2} - \sin^2\alpha\right)^{1/2},$$

$$1 = RP + RS + TP + TS. \tag{38}$$

For angles of incidence between the critical angles for P and S, the condition for continuity of the instantaneous flux becomes

$$\sin^2\phi = A_{1P}^2 \sin^2(\phi + \phi_{1P}) + A_{1S}^2 \frac{\cos\alpha}{\sin^2\alpha}\left(\frac{V_1^2}{v_1^2} - \sin^2\alpha\right)^{1/2} \sin^2(\phi + \phi_{1S})$$

$$+ A_{2P}^2 \frac{\rho_2}{\rho_1} \frac{\cos\alpha}{\left(\sin^2\alpha - \frac{V_1^2}{V_2^2}\right)^{1/2}} \sin(\phi + \phi_{2P})\cos(\phi + \phi_{2P}) \tag{39}$$

$$+ A_{2S}^2 \frac{\rho_2}{\rho_1} \frac{\cos\alpha}{\sin^2\alpha}\left(\frac{V_1^2}{v_2^2} - \sin^2\alpha\right)^{1/2} \sin^2(\phi + \phi_{2S}), \quad \alpha_P < \alpha < \alpha_S = \sin^{-1} V_1/v_2.$$

The $\phi_{ij}(\alpha)$ are variable from zero to 2π. The time and x dependence enter into the flux for each wave in a different way. Consequently, Knott's equation is not the correct condition for continuity of the *instantaneous* flux when the angle of incidence exceeds the critical angle. To find the correct condition, we express the trigometric functions in terms of the double-angle formulas

$$\sin^2(\phi + \phi_{ij}) = \tfrac{1}{2}(1 - \cos 2(\phi + \phi_{ij})) = \tfrac{1}{2}(1 - \cos 2\phi \cos 2\phi_{ij} + \sin 2\phi \sin 2\phi_{ij}),$$

$$\sin(\phi + \phi_{ij}) \cos(\phi + \phi_{ij}) = \tfrac{1}{2} \sin 2(\phi + \phi_{ij}) = \tfrac{1}{2}(\sin 2\phi \cos 2\phi_{ij} + \sin 2\phi_{ij} \cos 2\phi),$$

and rewrite equation (39) in the form

$$(1)C_1 + (\cos 2\phi)C_2 + (\sin 2\phi)C_3 = 0. \tag{40}$$

This equation must be satisfied for all values of time and x. The functions 1, sin 2ϕ, cos 2ϕ are linearly independent. Therefore, equation (40) cannot be satisfied unless $C_1 = C_2 = C_3 = 0$. Writing out these equations gives

$$C_1 = 0, \; 1 = RP + RS + TS, \tag{41}$$

$$C_2 = 0, \; 1 = A_{1P}^2 \cos 2\phi_{1P} + A_{1S}^2 \frac{\cos\alpha}{\sin^2\alpha}\left(\frac{V_1^2}{v_1^2} - \sin^2\alpha\right)^{1/2} \cos 2\phi_{1S} \tag{42}$$

$$- \frac{\rho_2}{\rho_1} A_{2P}^2 \frac{\cos\alpha}{\left(\sin^2\alpha - \frac{V_1^2}{V_2^2}\right)^{1/2}} \sin 2\phi_{2P} + \frac{\rho_2}{\rho_1} A_{2S}^2 \frac{\cos\alpha}{\sin^2\alpha}\left(\frac{V_1^2}{v_2^2} - \sin^2\alpha\right)^{1/2} \cos 2\phi_{2S},$$

$$C_3 = 0, \; 0 = A_{1P}^2 \sin 2\phi_{1P} + A_{1S}^2 \sin 2\phi_{1S} \frac{\cos\alpha}{\sin^2\alpha}\left(\frac{V_1^2}{v_1^2} - \sin^2\alpha\right)^{1/2} \tag{43}$$

$$+ \frac{\rho_2}{\rho_1} A_{2P}^2 \cos 2\phi_{2P} \frac{\cos\alpha}{\left(\sin^2\alpha - \frac{V_1^2}{V_2^2}\right)^{1/2}} + \frac{\rho_2}{\rho_1} A_{2S}^2 \sin 2\phi_{2S} \frac{\cos\alpha}{\sin^2\alpha} \cdot \left(\frac{V_1^2}{v_2^2} - \sin^2\alpha\right)^{1/2}.$$

Inside the critical angle for P, Knott's equation insures continuity of the instantaneous flux. Intermediate between the critical angles Knott's equation must be replaced by the three equations (41) to (43). To interpret these equations, we proceed as follows. By integrating each term in equation (39) over a period, we obtain the net flux or total energy carried away from the interface by each wave type in one period. Intermediate between the two critical angles, the net flux in the transmitted P wave vanishes. Equation (41) is just the condition for continuity of the net flux. The individual terms on the right side of equation (41) determine the ratios of the net flux in the reflected and transmitted waves to the net flux in the incident wave. The difference in the physical significance attached to the terms in Knott's equation and to the terms in equation (41) is important. In Knott's equation the terms determine the ratio of the instantaneous flux and the net flux (because they are identical) whereas in equation (41) the terms determine the ratio of the net flux only. If in Knott's equation we replace the amplitude factors (A_{ij}) by the reflection and transmission coefficients [equation (11)] and use equations (7)–(9) to define the square roots beyond the critical angle, we obtain a complex equation. The real part of this equation gives equation (42) and the imaginary part gives equation (43). *Therefore, beyond the critical angle, continuity of the net flux and satisfaction of the real and imaginary parts of Knott's complex equation are required to insure continuity of the instantaneous flux.* Inside the critical angle Knott's equation insures continuity of both the instantaneous and net flux.

Beyond the critical angle for S, the equation of continuity for the net flux is

$$1 = RP + RS, \quad \alpha > \alpha_S. \tag{44}$$

The net flux in the transmitted P and S waves vanishes and the net flux in the incident wave is partitioned between the reflected waves. Equation (44) and the two equations obtained from Knott's complex equation insure continuity of the instantaneous flux.

We have computed the quantities

$$RP = A_{1P}^2,$$

$$RS = A_{1S}^2 \frac{\cos\alpha}{\sin^2\alpha}\left(\frac{V_1^2}{v_1^2} - \sin^2\alpha\right)^{1/2},$$

$$TP = A_{2P}^2 \frac{\rho_2}{\rho_1} \frac{\cos\alpha}{\left(\left|\dfrac{V_1^2}{V_2^2} - \sin^2\alpha\right|\right)^{1/2}},$$

$$TS = A_{2S}^2 \frac{\rho_2}{\rho_1} \frac{\cos\alpha}{\sin^2\alpha}\left(\left|\frac{V_1^2}{v_2^2} - \sin^2\alpha\right|\right)^{1/2}.$$

Inside the critical angle for P, each quantity may be interpreted either as the ratio of the instantaneous flux or as the ratio of the net flux. Intermediate between the two critical angles RP, RS, and TS determine the net flux ratios but not the instantaneous flux ratios. $TP/2$ is the ratio of the instantaneous flux maxima—(transmitted P)/(incident P). These maxima are not attained at the same time. Beyond the critical angle for S the quantities RP and RS determine the net flux ratios. $TP/2$ and $TS/2$ determine the ratios of the instantaneous flux maxima.

REPRESENTATIVE CASES

Our study of the data presented in the tables has not been as extensive as the amount of data warrants. Here we present only the results for a few representative cases. We feel that there is much more information to be extracted from these data.

Figure 3 is typical of cases in which there is no critical angle. The energy coefficient for the transmitted P wave (TP) is a slowly decreasing function of the angle of incidence. Even at 50 degrees, TP carries about 75 percent of the incident energy. As the angle of incidence approaches 90 degrees, TP decreases rapidly to zero. Except when the density or velocity contrast is large, TP carries away most of the incident energy within the critical angle.

In Figures 4 and 5 the velocity ratio is unity. A

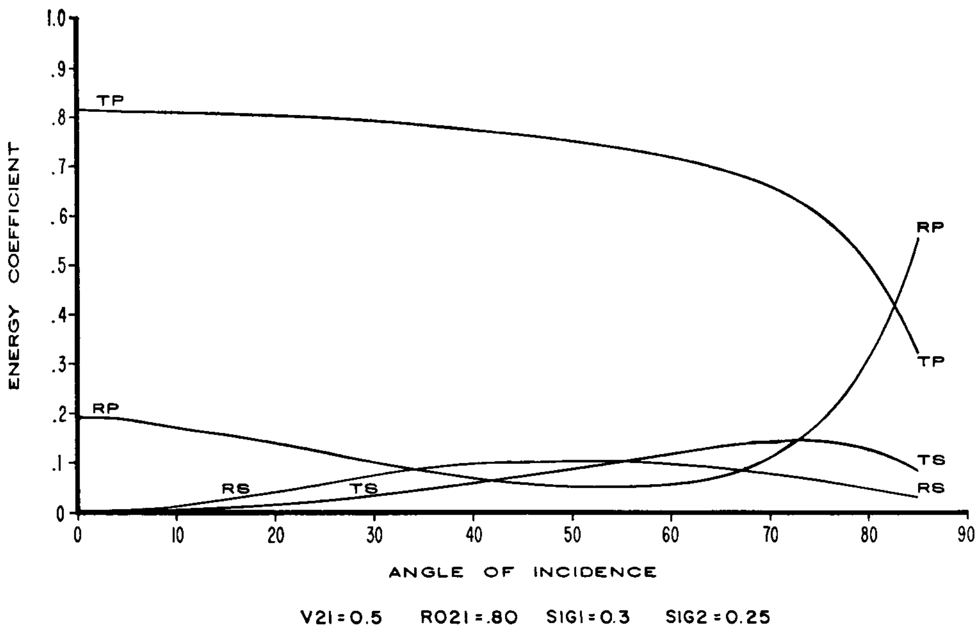

FIG. 3. Energy ratios for a case where there is no critical angle for the transmitted P wave.

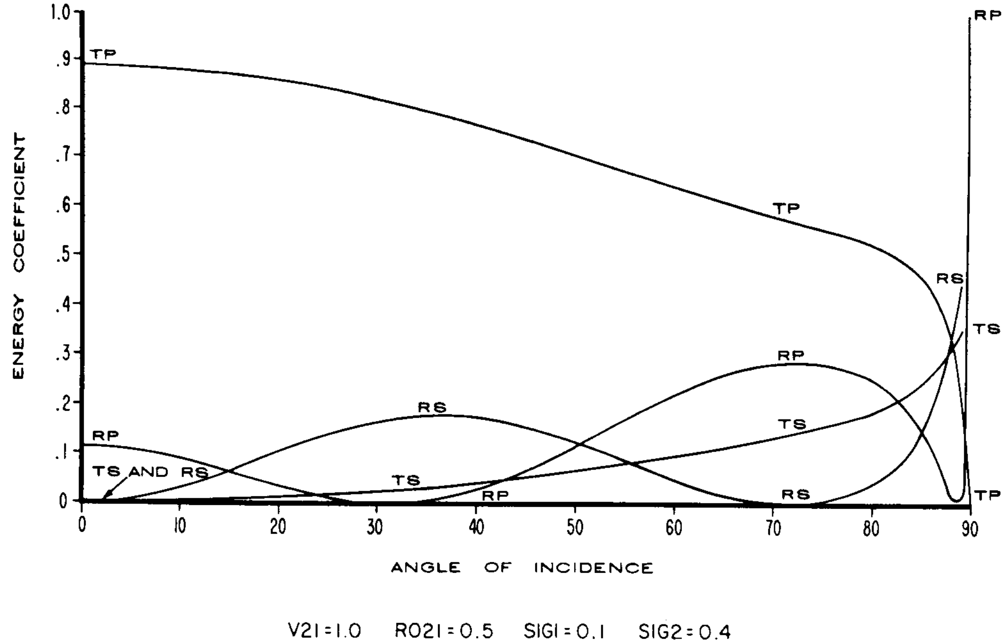

FIG. 4. Energy ratios for a case where $V_1/V_2 = 1.0(\alpha_P = 90°)$.

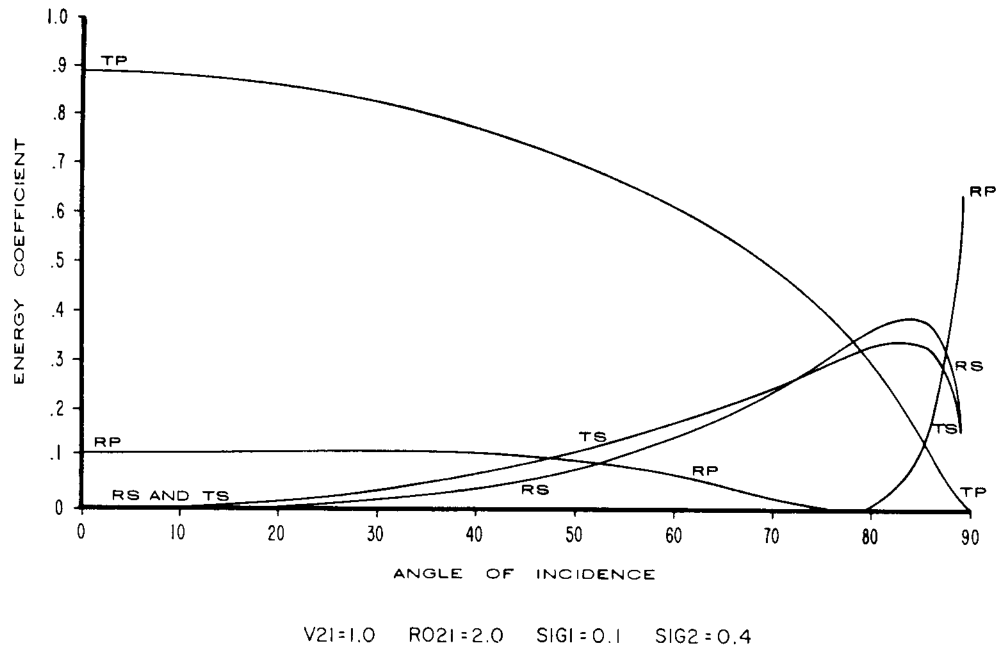

FIG. 5. Energy ratios for a case where $V_1/V_2 = 1.0(\alpha_P = 90°)$.

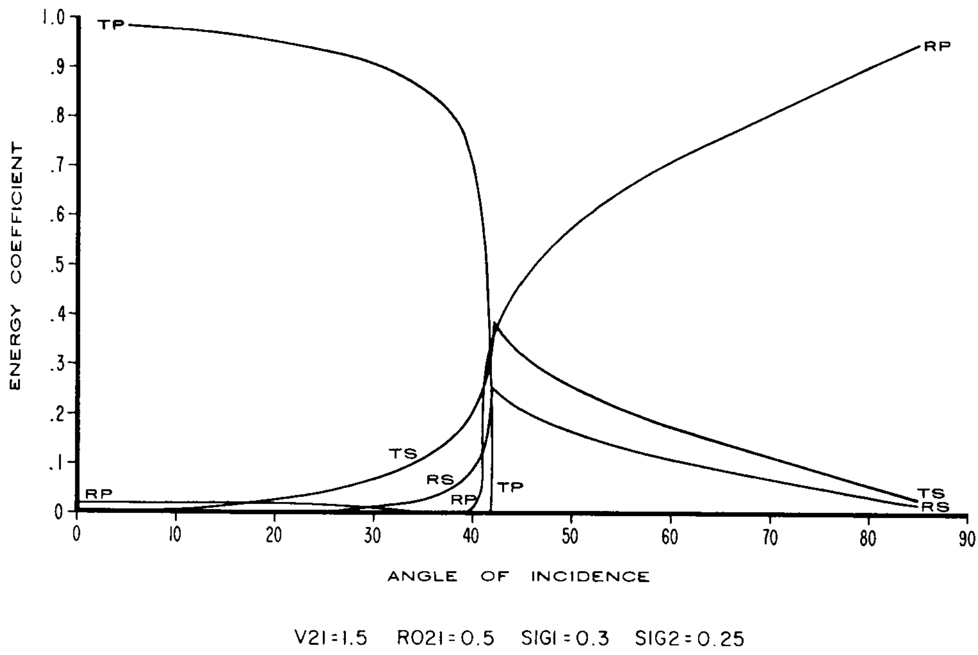

FIG. 6. Energy ratios for a case where $\alpha_P < 90°$.

significant difference between these cases and Figure 3 is the sharp maximum in the S-wave energy near grazing incidence. 90 degrees is actually thc critial P angle and the various energy coefficients behave much as they do just inside a critical angle. In Figure 4, RS exceeds RP even at relatively small angles of incidence.

Figure 6 is typical of cases in which there is a critical P angle. These cases are characterized by an abrupt change in slope at the critical angle. The transmitted P energy decreases slightly as the angle of incidence increases. Then it begins to decrease abruptly just before the critical angle and is zero at and beyond the critical angle. Beyond the critical P angle, the instantaneous energy flux in the transmitted P wave oscillates in direction and averages to zero when integrated over a period. The reflected P energy is, in general, small within the critical angle, but it increases to large values near the critical angle, and under certain conditions there is perfect reflection with no conversion and no transmission.

In Figure 7, the transmitted P wave and the transmitted S wave both have critical angles. Beyond the S-critical angle the net flux in the transmitted shear wave is zero and there is no net flux of energy directed into the high-velocity medium.

Figures 8 through 11 show the effect on RP of varying each parameter. The case $V21=1.5$, $R021=1.0$, $SIG1=SIG2=0.25$ is common to each figure. In Figure 8, $R021$ is varied over a sampling of the $R021$ values in the tables while the remaining parameters remain fixed. Figure 9 shows the influence of varying $SIG1$. Note the very sharp peak when $SIG1=0.5$. Figure 10 shows the effect of varying $SIG2$. For $SIG2=0.4$, we have almost a perfect P-wave reflector beyond the critical angle. Figure 11 depicts the influence of varying $V21$ which, of course, varies the critical angle and gives this figure its complex appearance.

In situations where one medium is a fluid and the other a solid, large amounts of S energy are generated in the solid medium at large angles of incidence by a P-wave incident from either medium. Ergin's (1952) figures show this phenomena. In many cases, this conversion has an efficiency of better than 50 percent for some tens of degrees of angle of incidence. Figures 12 through 14 show the efficiency of this conversion when the second

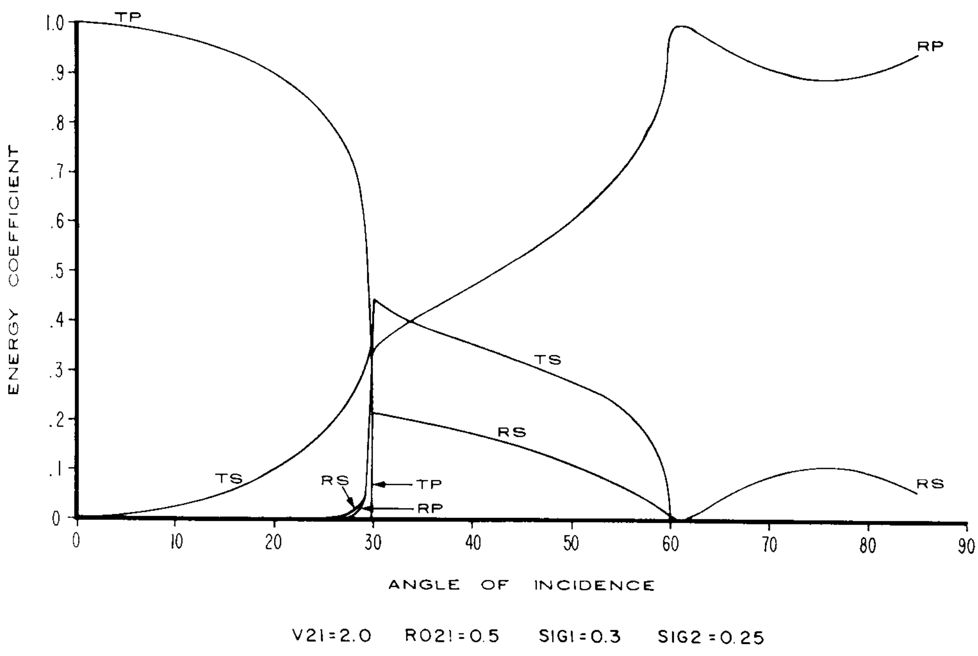

FIG. 7. Energy ratios for a case where $\alpha_P < \alpha_S < 90°$.

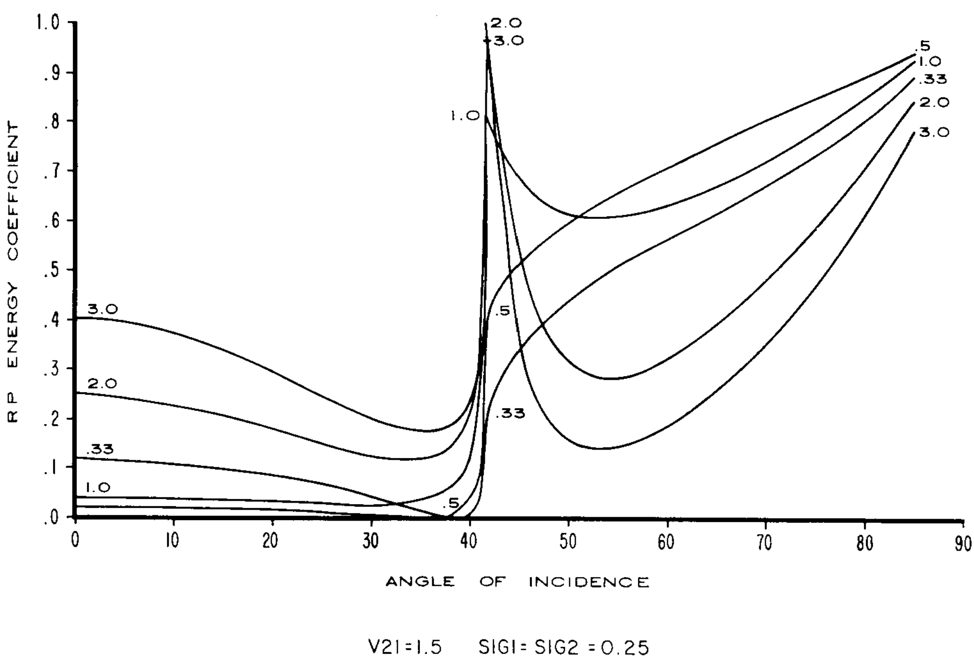

FIG. 8. The effect on the reflected compressional energy of varying the density ratio.

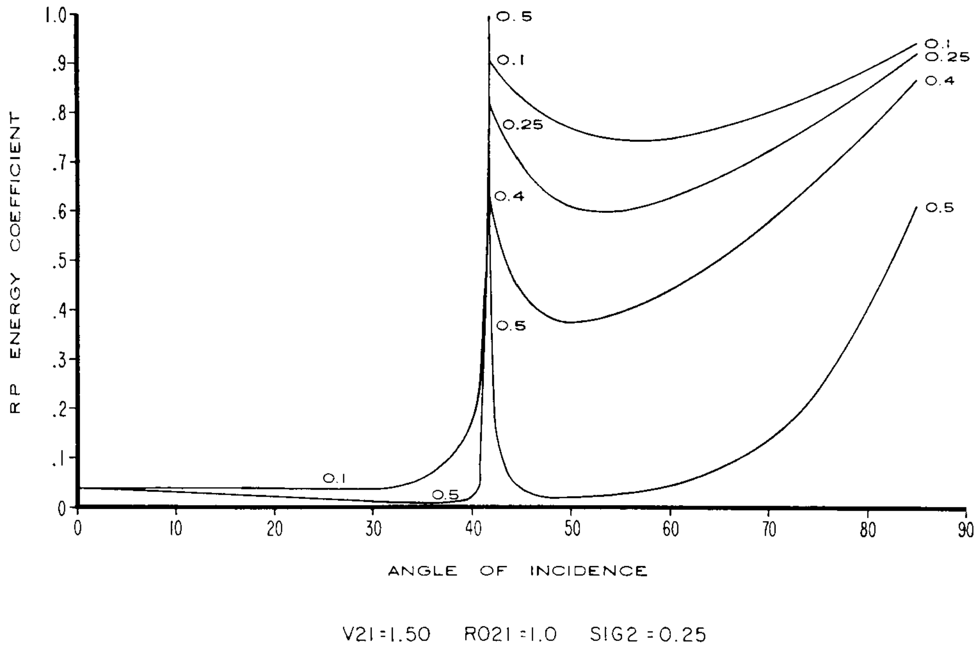

FIG. 9. The effect on the reflected compressional energy of varying the Poisson's ratio of medium 1.

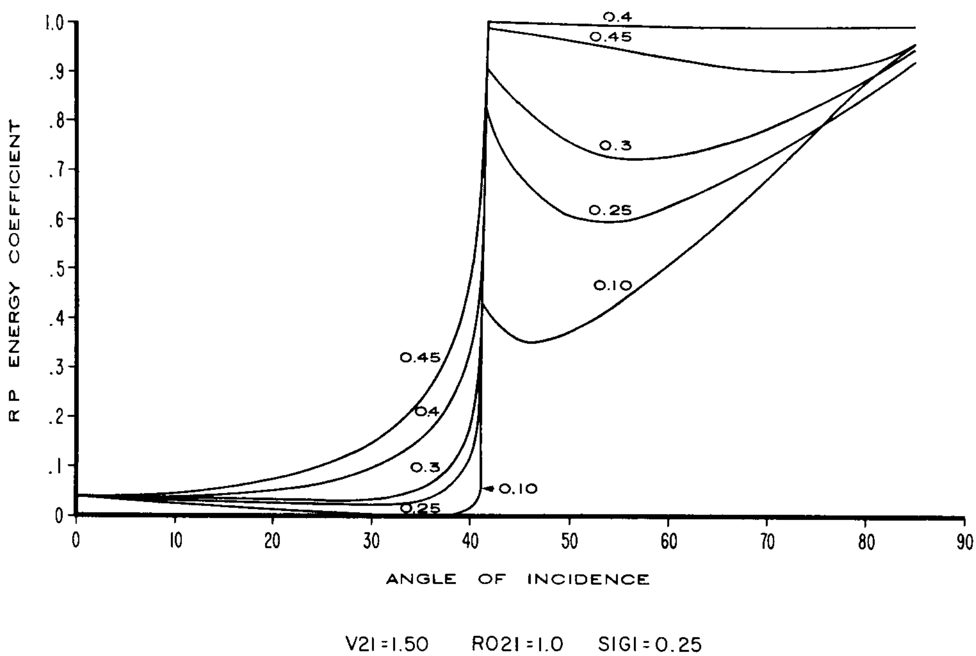

FIG. 10. The effect on the reflected compressional energy of varying the Poisson's ratio of medium 2.

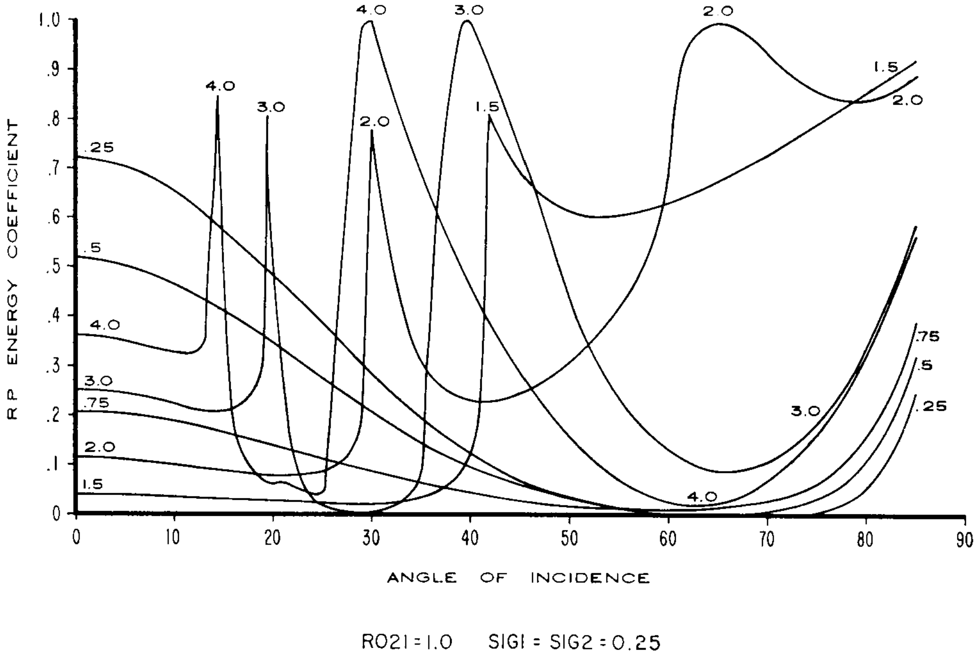

FIG. 11. The effect on the reflected compressional energy of varying the compressional velocity ratio.

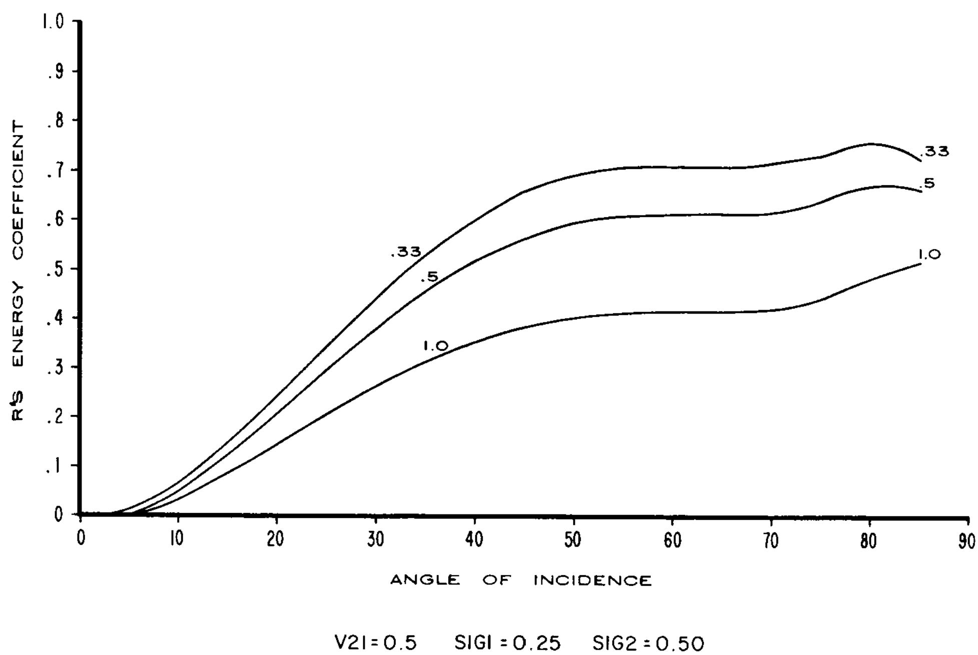

FIG. 12. The effect of varying the density ratio on the shear energy reflected at a solid-liquid interface.

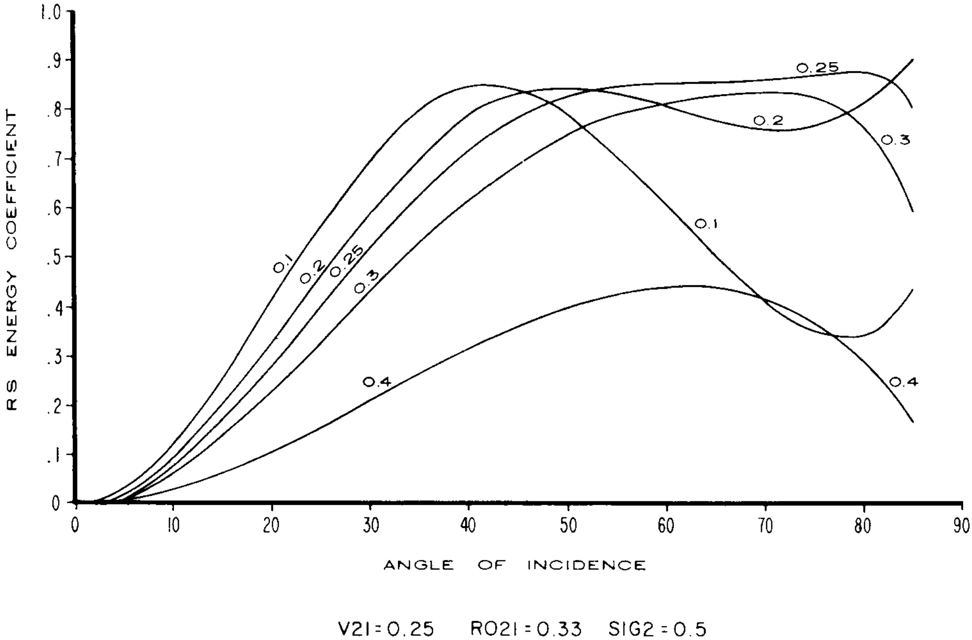

FIG. 13. The effect of varying the Poisson's ratio of medium 1 on the shear energy reflected at a solid-liquid interface.

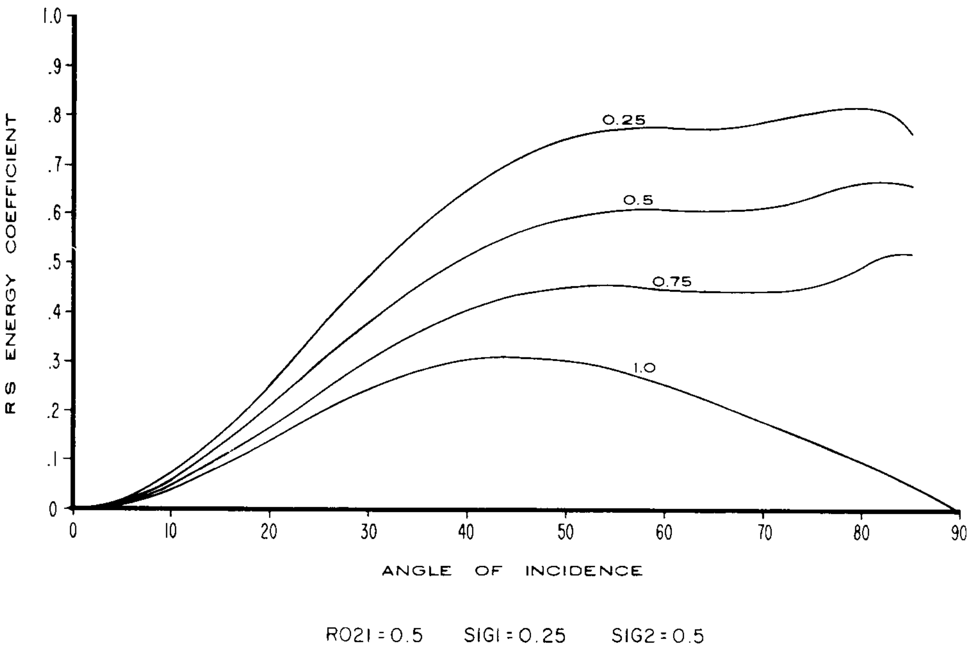

FIG. 14. The effect of varying the compressional velocity ratio on the shear energy reflected at a solid-liquid interface.

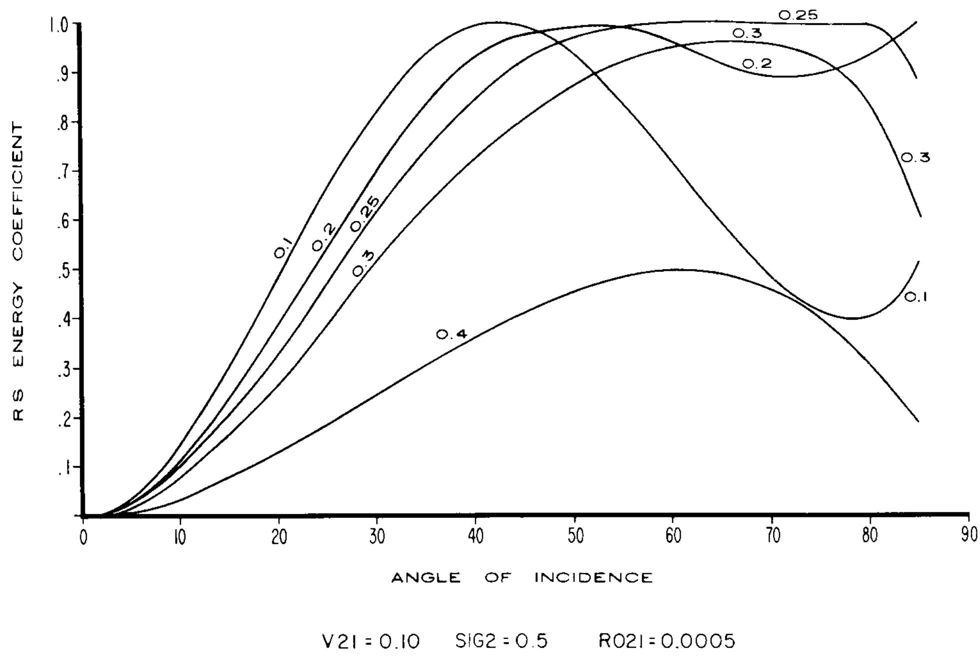

FIG. 15. The effect of varying the Poisson's ratio of medium 1 on the shear energy reflected from a solid-air interface.

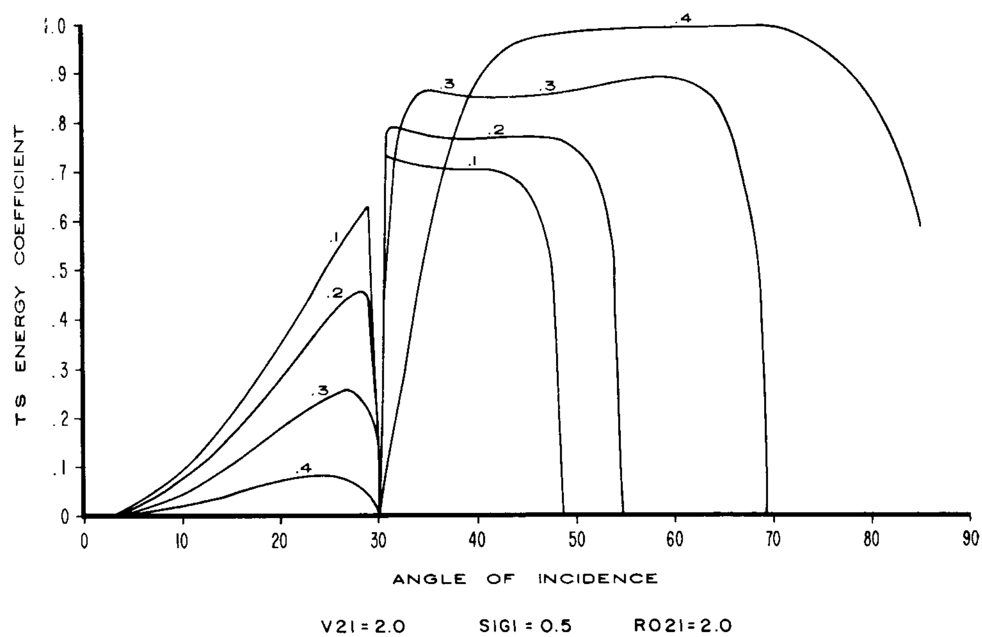

FIG. 16. The effect of varying the Poisson's ratio of medium 2 on the shear energy generated at a liquid-solid interface.

medium is a fluid and how the efficiency is altered by changing the parameters.

Figure 15 shows how much of the incident energy is converted to S energy upon reflection from a solid-air interface. Note that when $SIG1 = 0.25$, the efficiency of the conversion is above 95 percent for angles of incidence from 42 to 83 degrees.

Figure 16 shows the transmitted S-wave energy coefficient for several cases where a P wave is incident from a liquid to a solid. Again we have a very efficient S-wave generator with 95 percent of the incident energy going into the transmitted S wave for angles of incidence from 43 to 75 degrees when $SIG2 = 0.4$.

CONCLUSIONS

We have studied the way in which the energy in a plane P wave is partitioned among the reflected and transmitted waves at a plane interface. Extensive tables have been computed which show how the energy-flux ratios and relative phases in the vertical displacements depend upon the compressional velocity ratio, the Poisson's ratios, and the density ratio. An attempt has been made to cover a sufficiently large range of the parameters so that the results will find application in model seismology and velocity measurements as well as in geophysical investigations. The computations were performed for two solids, solid and air, solid and fluid, and two fluids. Computations for the solid-fluid and solid-air cases show that a large fraction of the incident energy goes into the shear wave over a wide range of angles of incidence.

Beyond the critical angle, Knott's equation is not the correct expression for continuity of the instantaneous flux. Beyond the critical angle for a particular wave, the flux periodically reverses direction and the net flux is zero. The individual terms in Knott's equation determine both the instantaneous and net flux ratios inside the critical angle (where they are identical) but only the net flux ratios beyond the critical angle (where the waves are out-of-phase). To achieve continuity of the instantaneous flux beyond the critical angle, three equations must be satisfied. Beyond both critical angles, the instantaneous flux in the individual transmitted waves may be quite large. When this happens, the individual components are nearly 180 degrees out-of-phase so that the sum cannot exceed the flux in the incident wave.

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