Applied Geophysics – GEO 5660/6660
Homework Assignment # 1
<u>Due: January 28th, 3 PM</u>

1. In class we obtained the velocities of P- and S-waves, V_P, V_S, directly from the proportionality constant in the dilatational and shear "components" of the wave equation. These velocities were expressed in terms of the Lamé constants, λ, & μ (shear modulus).
Given:

$$V_P=\sqrt{\frac{\lambda+2\mu}{\rho}}=\sqrt{\frac{K+\frac{4}{3}\mu}{\rho}}, \qquad V_S=\sqrt{\frac{\mu}{\rho}}, \qquad \nu=\frac{\lambda}{2(\lambda+\mu)}$$

where ρ is the density, K, the bulk modulus, and ν, the Poisson's ratio,

(a) Express K in terms of λ & μ.

(b) Express μ in terms of K & ν alone.

(c) For an incompressible fluid, which deforms without resistance to shear stress, $\mu = 0$. What value of Poisson's ratio does this correspond to? This would be the maximum theoretical value of ν.

(d) Express λ & μ in terms of ρ, V_P & V_S.

(e) Express the V_P/V_S ratio in terms of ν alone, showing that this ratio is independent of density. What is this ratio for an incompressible fluid, and what are the implications for shear velocity, V_S?

(f) Plot V_P/V_S ratio as a function of ν between 0.05 to 0.5 (*i.e.*, over an order of magnitude range). What would be the V_P/V_S ratio for a material having a Poisson's ratio, ν= 0.25, typically assumed for "rocks"?

2. Monochromatic plane waves: Consider the following waves propagating in the x direction in a uniform isotropic medium:

(a) P-wave in which $u_x = A\sin(\omega t-kx)$, and

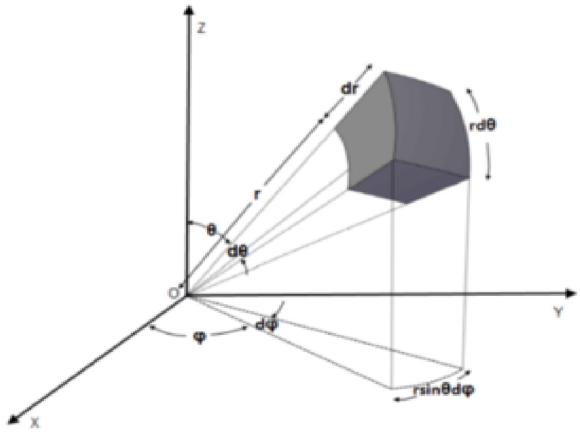
(b) S-wave with displacements only in the y direction, *i.e.*, $u_y = A\sin(\omega t-kx)$.

For each case, derive expressions for the nonzero components of the (2D) stress tensor, given:

$$e=\begin{bmatrix}\frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)\\ \frac{1}{2}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right) & \frac{\partial u_y}{\partial y}\end{bmatrix} \qquad \sigma=\begin{bmatrix}\lambda\theta+2\mu e_{xx} & 2\mu e_{xy}\\ 2\mu e_{xy} & \lambda\theta+2\mu e_{yy}\end{bmatrix}, \quad \textit{where} \quad \theta=(e_{xx}+e_{yy})$$

(c) If for the above P-waves, V_P= 10 km/s, and the maximum strain is 10^{-8}, what is the maximum particle displacement for waves with periods, T, of: *(i)* 1 s, *(ii)* 10 s, *(iii)* 100 s? Recall that angular frequency, $\omega = 2\pi/T$. If the density of the medium, $\rho = 2800\ kg/m^3$, what is the average kinetic, potential and total energy density within the P-wave field at each period? Refer to lecture-board images posted on the course webpage.

3. 1D Spherical Wave Equation & General Solution: Derive the 1D spherical wave equation show that its general solution has a (*1/r*) dependence, as outlined in class (and steps below). Refer to lecture-board image on the course webpage.



(a) Set up the force balance (Newton's Second Law), ignoring body forces (this is a good approximation except for long-period normal modes). Show that ΣF = ma results in:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sigma_{rr}\right)=\rho\frac{\partial^2 u}{\partial t^2}$$

(b) Now introduce the 1D stress-strain relationship, $\sigma_{rr} = (\lambda + 2\mu)\partial u/\partial r$, to obtain:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

(c) Now, show that:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)=\frac{1}{r}\frac{\partial^2(r u)}{\partial r^2}$$

(d) Plugging the right hand side of this equality into (b), then multiplying throughout by *r*, to obtain:

$$\frac{\partial^2(r u)}{\partial r^2}=\frac{1}{c^2}\frac{\partial^2(r u)}{\partial t^2}$$

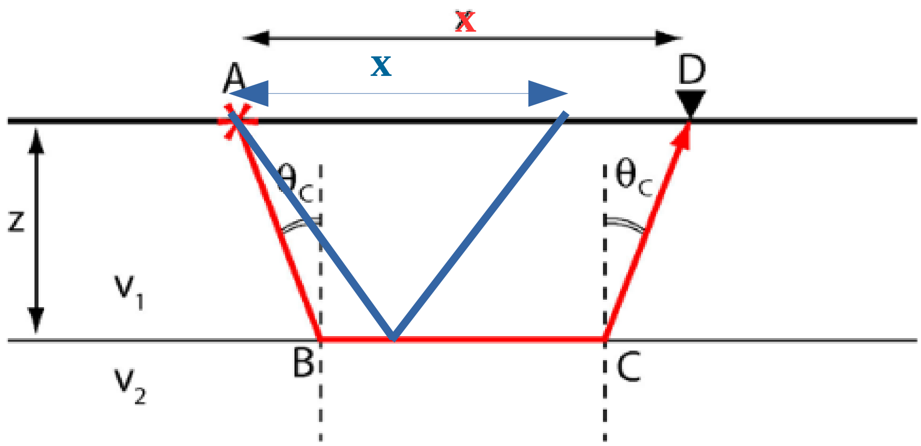
(e) Finally, based on the general solutions to the cartesian wave equation discussed in class, and considering only the OUTWARD propagating waves (+*r* direction), the general solution will be:

$$(r u)=f\,(kr-\omega t)$$

and therefore,

$$u=\frac{f\,(kr-\omega t)}{r}$$

4. Two layer reflection & refraction: Consider P-waves traveling through a two layer (each uniform, and isotropic) Earth model with the surface layer of thickness z as shown, and $v_2 > v_1$. The P-wave incident at a critical angle is refracted along the interface (*e.g.*, red ray) between the two layers. Since the wave is refracted towards the horizontal. As the angle of incidence is increased, the geometry results in a head wave travelling horizontally in layer 2. Rays with larger incidence angles are completely reflected back to the surface (*e.g.*, blue ray). Using Snell's Law, the fact that travel time is path length over velocity in a given layer, and a little geometry/trigonometry, derive the travel time as a function of surface location, x for the following waves:



(a) Direct Wave: This relationship will be linear in x, with a slope of $1/v_1$. Recall that the direct wave travels along the surface at a velocity of v_1.

(b) Reflected Wave: This relationship will be non-linear with respect to x.

(c) Refracted Wave: This relationship will also be linear in x, with a slope of $1/v_2$. This slope can be used to calculate the velocity of layer 2. Make use of the fact that: $Sin(\theta_c) = v_1/v_2$.

In all cases, you should be able to express the t-x entirely in terms of x,v_1, v_2, and z.

(d) For $z = 30$ *km*, $v_1 = 5$ *km/s*, and $v_2 = 8$ *km/s*, plot each of the above *t-x* curves on the same graph with time axis going from 0-60 *s*, and x-axis from 0-320 *km* (~ 200 *miles*). If everything above was correct, the curve for the reflected ray should plot above the other two curves (i.e., reflected wave arrivals are always later than the direct and refracted rays), and should asymptotically approact the direct wave for large distances from the source **(WHY?)**. You can use a spread-sheet or a programming/plotting language of your choice.

For (e)-(f), first derive the analytical expression, then substitute values for the appropriate independent variables to check against this *t-x* graph.

(e) $x_{critical}$: This is the minimum distance beyond which a refracted wave is observed at the surface. Determine $x_{critical}$ by setting the refracted path, *BC* to zero. The result should depend only on layer depth, z, and critical angle, θ_c. For above parameters, check that this expression yields plotted $x_{critical}$.

(f) $x_{crossver}$: The location beyond which the refracted wave is the first observed arrival (i.e., ahead of the direct wave). Derive the expression for this distance by equating $t_{direct}(x)$ and $t_{refraction}(x)$. The result should depend only on layer depth, z, and velocities v_1 & v_2, only. For above parameters, check that this expression yields plotted $x_{crossover}$.

(g) Layer depth, z: The layer depth can be computed by extrapolating the linear *t-x* relationship for the refracted ray to the source ($x=0$). Derive an expression for the t-intercept, t_i, from $t_{refraction}(x=0)$, then rearrange to express depth in terms of t_i, and velocities v_1 & v_2, only. For above parameters, check that this expression yields plotted t_i. Or use the plotted t_i to verify the layer depth, z.