Gravity Method

Burger 349-425 (Ch. 6)

Conservative Fields: Gravity, Electrostatic, and Magnetic fields follow the ***Poisson's equation*** of the form:

$$\nabla^2\psi = f\,(\text{sources})$$

where ψ is a ***potential***, $\vec{\nabla}$ is the gradient operator:

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Notation: Here, the arrow $\rightarrow$ denotes a vector quantity;
the carat ^ denotes a unit direction vector.

Hence, the gradient operator is just a vector form of slope…

Because ***Poisson's*** eqn. always incorporates a potential ψ, we call these ***Potential Field Methods***

Potential Field Theory

$$For\ any\ \ CONSERVATIVE\ \ field\,,\ \ F: \qquad \vec{F} = \vec{\nabla}\psi \qquad (1)$$

$$Fluxes: Johann\ Carl\ Friedrich\ Gauss\,(1777-1855):$$

$$Gauss'\ Law\ for\ gravity: \quad \Phi_g = \oiint_S \vec{g}\cdot\vec{n}\,dA = -4\pi GM \qquad (2a)$$

$$Gauss'\ Law\ for\ electrostatics: \quad \Phi_E = \oiint_S \vec{E}\cdot\vec{n}\,dA = \frac{q}{\epsilon_0} \qquad (2b)$$

$$Gauss'\ Law\ for\ magnetism: \quad \Phi_B = \oiint_S \vec{B}\cdot\vec{n}\,dA = 0 \qquad (2c)$$

$$But\ by\ Gauss'\ Divergence\ Theorem\ \ (for\ gravity):$$

$$\Phi_g = \oiint_S \vec{g}\cdot\vec{n}\,dA = \oiiint_V \vec{\nabla}\cdot\vec{g}\,dV \qquad (3)$$

$$Also, \qquad M = \oiiint_V \rho\,dV \qquad (4)$$

Potential Field Theory

Thus, $(2a)$, (3), (4) *yield*:

$$\Phi_g = \oiiint_V \vec{\nabla}\cdot\vec{g}\, dV = -4\pi GM = -4\pi G \oiiint_V \rho\, dV \qquad (5)$$

Therefore, $\vec{\nabla}\cdot\vec{g} = -4\pi G\rho \qquad (6)$

But $(1) \Rightarrow \vec{g} = \vec{\nabla}\psi$, *yielding*: $\qquad (7)$

$$\vec{\nabla}\cdot(\vec{\nabla}\psi) = -4\pi G\rho = \Im \ (sources) \qquad (8)$$

that is,

$$\nabla^2\psi = -4\pi G\rho = \Im \ , \quad \text{the POISSON's equation} \qquad (9)$$

which is similar for the electrostatic / magnetic fields.

Applying $(2a)$ *for a spherical mass distribution of uniform density* ρ, *radius*, r:

$$\oiint_S \vec{g}\cdot\vec{n}\, dA = g(4\pi r^2) = -4\pi GM$$

$$\Rightarrow \quad g = -\frac{GM}{r^2} \quad (\text{Newton's Law!}) \qquad (10)$$

Gravity

Gravitational field $\vec{g}$:

$$\vec{\nabla}\Psi = \vec{g}$$

Poisson's equation,

$$\nabla^2\Psi = \vec{\nabla} \bullet \vec{g} = -4\pi G \rho$$

given a mass density of ρ, where G is universal gravitational constant:

$$G = 6.672\text{x}10^{-11}\ \text{Nm}^2/\text{kg}^2$$

Gravitational Flux, Φ :

$$\text{Flux, } \Phi = -4\pi G \iiint_{Vol} \rho \, dV$$

Newton's Law of Gravitation

Thus, for a body with mass M is spherical with constant density:

$$\vec{g} = \hat{r}\,\frac{GM}{r^2}$$

Where, r is distance from the center of mass, and
$\hat{r}$ is the (unit) direction vector pointing toward the center.

Kepler's 3rd Law: $T^2 = kr^3$ (1619) → Newton's Law of Gravitation (1687):
Kepler's constant, $k \approx (3/4) \times 10^{-5}\ (\text{day})^2/(A.U.)^3 \approx 1\ (\text{yr})^2/(A.U.)^3$!!!
From Newton's Law, equating Centripetal & gravitational forces, $k = 4\pi^2/(GM)$

Now, $$\vec{F} = \hat{r}\,\frac{GMm}{r^2} = m\vec{g}$$

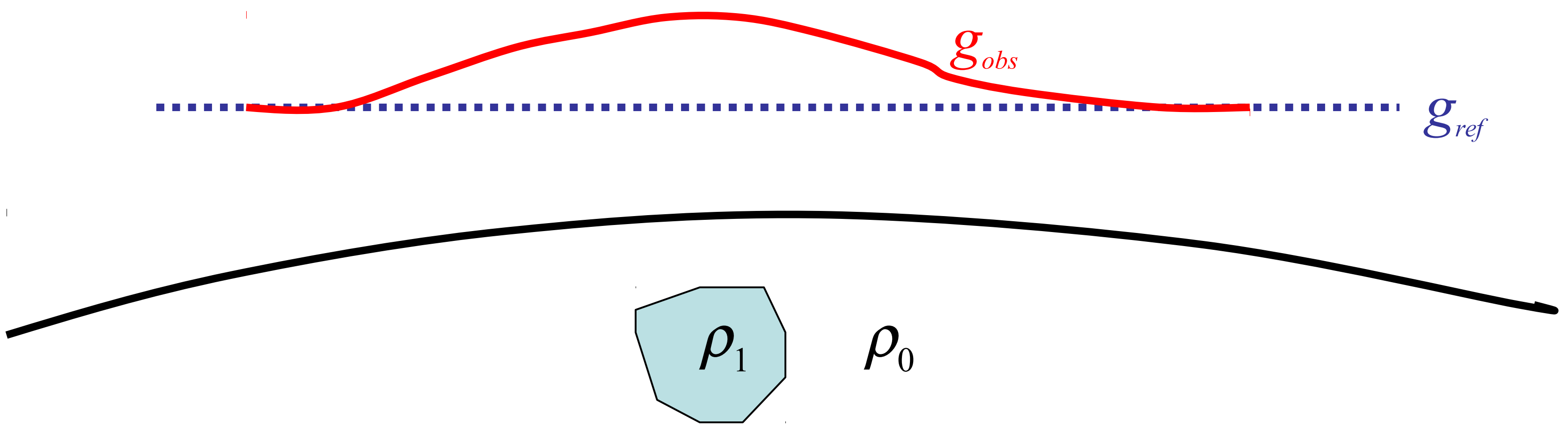
So $\vec{g}$ expresses the acceleration of m due to M, and has units of acceleration:
Gal in cgs (= 0.01 m/s² ≈ 1 *milli-**g***)

On the Earth's surface,

$$\vec{g} = \hat{r}\,\frac{GM_E}{R_E^2} = \hat{r}\,\frac{\left(6.67\times10^{-11}\right)\left(5.97\times10^{24}\right)}{\left(6.38\times10^{6}\right)^2} = 9.8\ m/s^2$$

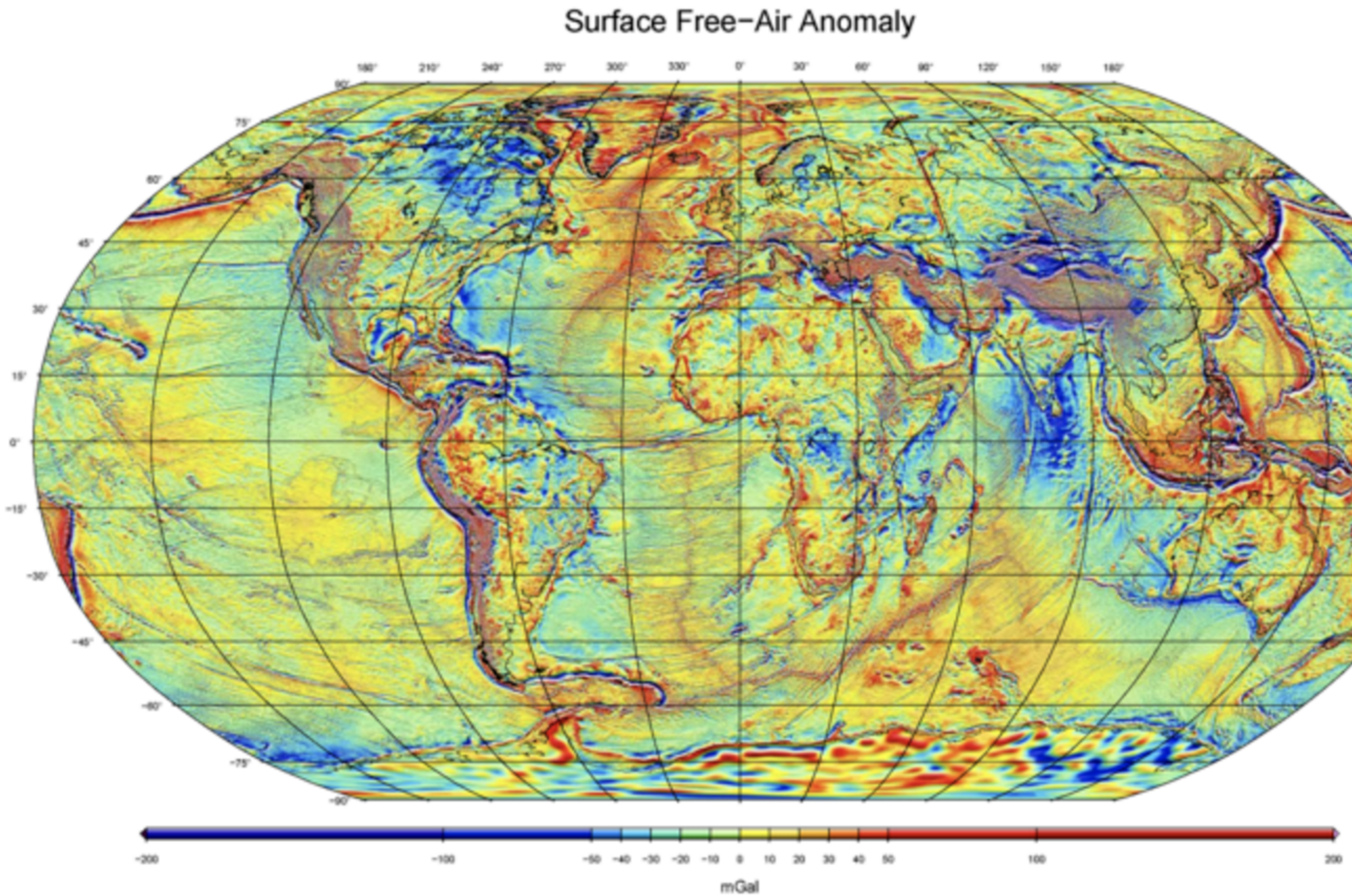
HOWEVER, ρ is ***not*** radially symmetric in the Earth… so $\vec{g}$ is neither constant or uniform!

Gravity methods look for ***anomalies***, or perturbations, from a reference value of $\vec{g}$ at the Earth's surface:



Example:

Global Free-Air Gravity Field from GRACE + GOCE + satellite altimetry + surface measurements…



WGM-2012 model from Bureau Gravimetríque International

Gravity Anomalies from GRACE